Coherence resonance in excitable and oscillatory systems: The essential role of slow and fast dynamics

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Stochastic noise of an appropriate amplitude can maximize the coherence of the dynamics of certain types of excitable systems via a phenomenon known as coherence resonance (CR). In this paper we demonstrate, using a simple excitable system, the mechanism underlying the generation of CR. Using analytical expressions for the spectral density of the system's dynamics, we show that CR relies on the coexistence of fast and slow motions. We also show that the same mechanism of CR holds in the oscillatory regime, and we examine how CR depends on both the excitability of the system and the nonuniformity of the motion.
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The apparent counterintuitive ability of noise to increase the coherence of a dynamical system has recently received considerable attention. By coherence, we mean the correlation of the system time-course with itself. For instance, noise can induce the synchronization of coupled oscillators [1] and regular synchronized oscillations in networks of coupled excitable elements [2,3]. Another popular noise-related topic is stochastic resonance (SR), which is a phenomenon wherein the correlation between a weak input signal (typically, periodic) and the response of a nonlinear system is maximized by the presence of a particular, nonzero level of noise [4–6]. However, even in the absence of an input signal, noise can maximize the coherence of certain types of nonlinear dynamical systems. Indeed, noise can induce the appearance of additional peaks in the output power spectrum when a system is placed close to a bifurcation [7]. In the vicinity of a saddle-node bifurcation, an optimal level of noise can maximize the coherence of the dynamics of an excitable system [8]. A similar phenomenon, referred to as *coherence resonance* (CR), has recently been demonstrated in several nonlinear dynamical systems [9,10]. It was suggested in Ref. [11] that the CR reported in Ref. [8] could be related to the coexistence of slow and fast motions in the system dynamics. This idea was partially confirmed in Ref. [10], where it was shown that the coefficient

$$R=\sqrt{\mathrm{var}(T)}/\langle T\rangle, \tag{1}$$

where T is the duration of a system cycle, depends mainly on the slow motion for low noise levels and the fast motion for high noise levels.

In this paper we examine the effects of slow and fast motions on CR in a one-dimensional excitable system. The dynamics of this system is reduced to a series of pulses for which an analytical expression of the spectral density is obtained. A measure of the coherence of the dynamics is derived and, in contrast to Ref. [10], we independently examine the effects of slow and fast motions on the coherence. This leads to the first demonstration that CR is caused by the distinct sensitivities of the durations of the slow and fast motions to the input noise. We also show that the same effects hold for the system in the oscillatory regime, and we investigate how CR is dependent on both the ratio of the slow and fast motions and the distance to the bifurcation.

Here we consider a one-dimensional system with a piecewise linear periodic potential. The system dynamics in the presence of noise is described by

$$\dot{x}=f(x)+\sqrt{D}\xi(t), \tag{2}$$

$$f(x)=(1-a)\Theta[(\pi/2)-x]+b\Theta[x-(\pi/2)], \tag{3}$$

where Θ is the Heaviside function, x is the phase of the system $(0\leqslant x\leqslant 2\pi)$, $a>0$, $b>0$, ξ is Gaussian noise with zero mean and correlation $\langle\xi(t_1)\xi(t_2)\rangle=\delta(t_1-t_2)$, and D is the noise intensity. For the sake of simplicity variables and parameters are dimensionless. The parameter a sets the system's dynamical regime. If a is less than 1 (in the absence of noise), the regime is oscillatory. If a is greater than 1, the regime is excitable, i.e., when x is moved away from the attractor $(x=0)$ such that it exceeds the threshold value $\pi/2$, an excursion is produced. We first consider an excitable system having two different time scales, where $1-a=0.05$ and $b=2.05$.

To quantify the coherence of the dynamics obtained with Eq. (2), we define a coherence measure β based on the spectral density, as in Refs. [8,9],

$$\beta=H\omega_p/\Delta\omega, \tag{4}$$

where H is the peak value in the spectrum, ω_p is the peak frequency, and $\Delta\omega$ is the width of the peak. The width is defined here as $\Delta\omega=\omega_r-\omega_p$, where ω_r is the frequency greater than ω_p for which the spectral density is $S(\omega_r)=H\exp(-0.5)$. Figure 1 shows the coherence β obtained with Eq. (2) as a function of the noise amplitude D. It can be seen that β is maximized for $D\approx 0.9$. We define the apparent period T of the system as the first passage time (FPT) over 2π when the initial condition is $x=0$. For $x=0$, the potential has a large steep barrier; thus, we can set $x=0$ as a

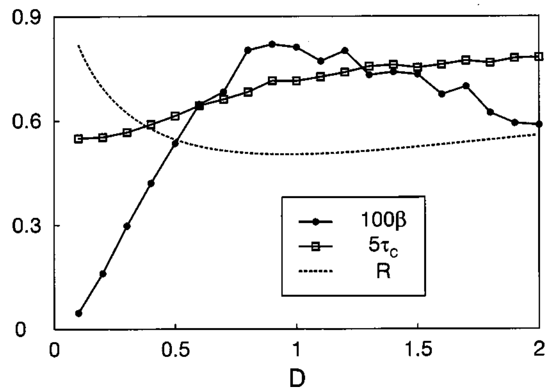

FIG. 1. Solid circles: coherence β of the dynamics of Eq. (2) as a function of the noise amplitude D. Each value of β is obtained by averaging over 500 power spectra of simulations lasting 33 000 time units with a time step $\delta t = 0.005$. Open boxes: characteristic correlation time τ_c of a series of pulses spaced by intervals corresponding to realizations of T. Dashed line: coefficient of variation R, given by Eq. (1).

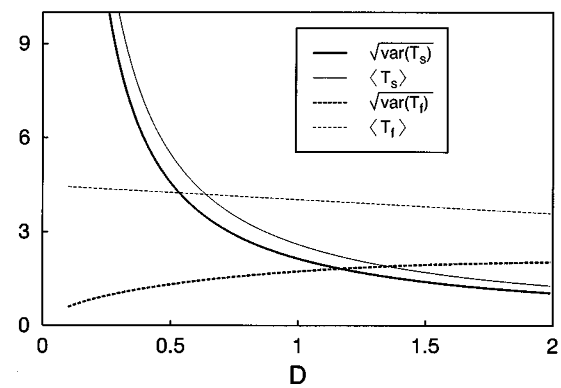

FIG. 2. Mean and square root of the variance of the duration T_s of slow motion and the duration T_f of fast motion as a function of the noise amplitude D.

reflecting barrier. With this convention, the first and second moments of the random variable T are given by [12]

$$\langle T\rangle = T_1(0) = \frac{2}{D}\int_0^{2\pi}\frac{dy}{\phi(y)}\int_0^y \phi z dz, \tag{5}$$

$$\langle T^2\rangle = \frac{4}{D}\int_0^{2\pi}\frac{dy}{\phi(y)}\int_0^y \phi z T_1(z) dz, \tag{6}$$

$$\phi(x) = \exp\left(\frac{2}{D}\int_0^x f(y)dy\right). \tag{7}$$

From these expressions the coefficient of variation R [Eq. (1)], used in Ref. [10] to quantify CR, can be obtained. Figure 1 shows R as a function of D. R is minimized for a value of D close to the one that maximizes the coherence β. If all the information required to account for CR was contained in R, then a signal having only T as source of variations should be able to reproduce CR. Such a simple signal $y(t)$ is a sequence of pulses $(y=1)$ of fixed duration l spaced by intervals $(y=0)$ having the probability distribution of T. This sequence can be simulated if one knows the probability distribution of T. Let $G(x,t)$ be the probability that the system Eq. (2) is still in the interval $[0;2\pi]$ at time t, if it was in $x\in[0;2\pi]$ at time $t=0$. Using the backward Fokker-Planck equation, one can show [12] that G obeys the following equation:

$$\partial_t G(x,t) = f(x)\partial_x G(x,t) + (D/2)\partial_x^2 G(x,t) \tag{8}$$

with auxiliary conditions

$$G(x,0) = 1(x\in[0;2\pi[), \tag{9}$$

$$G(2\pi,t)=0, \quad \partial_x G(0,t)=0. \tag{10}$$

Equation (8) can be simulated and the probability distribution function of T is then obtained by $P(t)=\mathrm{Prob}(T\leqslant t) = 1-G(0,t)$. Then, if U is a random variable in $[0;1]$ with uniform distribution, $P^{-1}(U)$ has the distribution of T. We simulated Eq. (8) and constructed series of pulses for different values of D. When we applied the coherence measure β to these series of pulses, we found that β was a monotonically increasing function of D. To show that this result is not related to the definition of β, we also used the characteristic correlation time $\tau_c=\int_0^{+\infty}C(t)^2dt$ (see Ref. [10]), where $C(t)$ is the normalized correlation coefficient for the time lag t. The boxes of Fig. 1 show that τ_c is a monotonically increasing function of D.

We thus conclude that to account for CR, at least two characteristic time scales of the system must be taken into account. This leads us to define two parts in the dynamics of the excitable system, namely, slow motion and fast motion. The definition of the durations of these parts in the presence of noise is complicated by the fact that the motion over $x = \pi/2$ is bidirectional. Accordingly, we adopt the following definition for the duration of the fast motion: T_f is the period of time that the system spent in the high-rate region $[f(x) = b]$ before crossing the value $x=2\pi$. This means that T_f is the FPT through 2π with initial condition $x=\pi/2$, provided $\pi/2$ is a reflecting barrier. The duration of the slow motion is thus defined as $T-T_f$. However, the duration of the slow motion can also be approximated by the FPT T_s through $\pi/2$, with the initial condition being the reflecting barrier $x = 0$. This can be done for two reasons. First, with a low noise level $(D<0.5)$, the probability density functions of T_s and $T-T_f$ differ little from one another and it is faster to compute the probability density function of T_s. Second, the mean and variance of both T_s and $T-T_f$ have the same dependence on D.

Figure 2 shows the variations of the mean and variance of T_s and T_f, respectively, as a function of D. As D increases, the means of both FPTs decrease, with $\langle T_s\rangle$ decreasing more sharply. It is further important to note that with increasing D, the fluctuations of T_s increase whereas those of T_f decrease. These opposite dependencies on D lead to the nonmonotonic noise-related behavior of the coherence β. To prove this assertion, we construct three types of signals and measure their coherence β as a function of the noise level. The first signal $y(t)$ is a sequence of pulses $(y=1)$ spaced by intervals $(y=0)$ that are realizations of T_f. To preserve the frequencies of the signal close to those of the system Eq. (2),

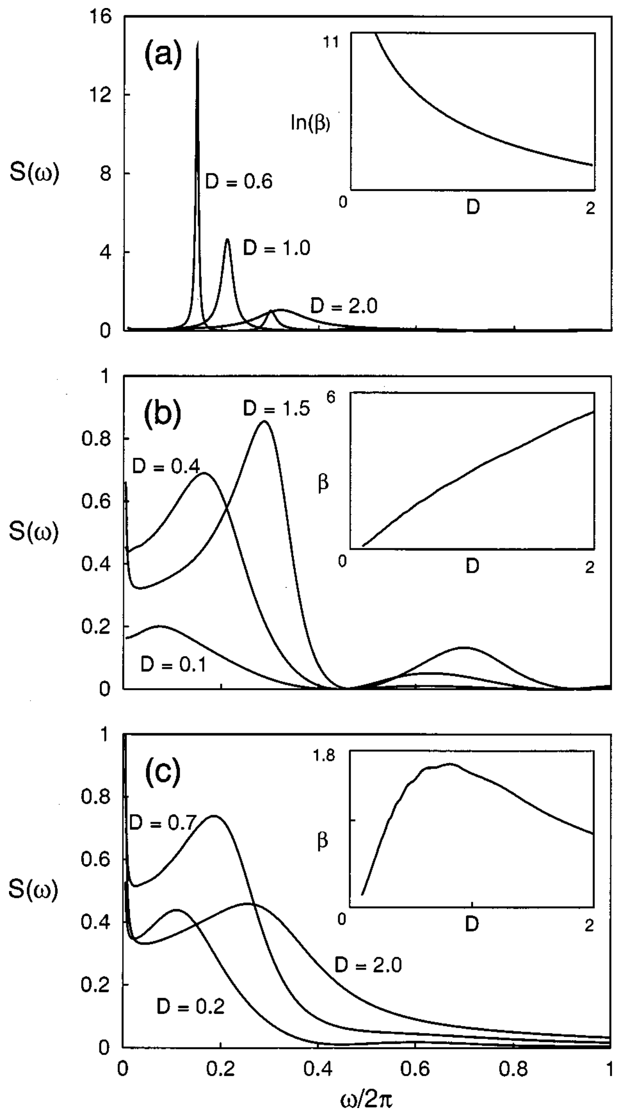

FIG. 3. Spectral densities of different pulse series: (a) only the duration T_f of the fast motion is variable, (b) only the duration T_s of the slow motion is variable, and (c) both T_f and T_s are variable. The insets show the coherence β as a function of the noise amplitude D.

the pulse duration is set equal to $\langle T_s\rangle$. The spectral density of this sequence of pulses, once the signal is centered, is given by [13]

$$S(\omega)=\frac{8\sin^2(\omega\langle T_s\rangle/2)}{\omega^2(\langle T_f\rangle+\langle T_s\rangle)}\frac{1-|\theta_f(\omega)|^2}{|1-\exp(i\omega\langle T_s\rangle)\theta_f(\omega)|^2}, \tag{11}$$

where ω is the frequency and $\theta_f(w)$ is the characteristic function of T_f. To determine θ_f, we simulated Eq. (8) with the left boundary $x=\pi/2$. Once the solution of Eq. (8) was computed, the probability density function w_f of T_s was obtained by $w_f(t)=\partial_t[1-G(\pi/2,t)]$. The real and imaginary parts of θ_f were then computed using w_f. Figure 3(a) presents the spectral densities obtained with $D=0.6$, 1.0, and 2.0. The height of the peak decreases with D much faster than the peak frequency increases, so that the coherence β decreases as shown in the inset. Thus, a signal with only T_f as a source of variation has a coherence that decreases with the noise level.

The second type of signal we constructed is a sequence of pulses where the pulse duration is set equal to $\langle T_f\rangle$ and the intervals between the pulses correspond to T_s. Figure 3(b) shows spectral densities obtained with this type of signal. It can be seen that as D increases, the coherence β is increased because the height of the peak increases and the width decreases.

The third type of signal we constructed is a sequence of pulses where the pulse durations are realizations of the random variable T_f and the intervals between them correspond to T_s. Figure 3(c) shows spectral densities obtained with this type of signal. In this case, the spectral density at frequency ω is given by [13]

$$S(\omega)=\frac{4\omega^{-2}}{\langle T_f\rangle+\langle T_s\rangle}\mathrm{Re}\left[\frac{[1-\theta_s(\omega)][1-\theta_f(\omega)]}{1-\theta_s(\omega)\theta_f(\omega)}\right], \tag{12}$$

where θ_s is the characteristic function of T_s and Re denotes the real part. When the variations of both characteristic times T_f and T_s are introduced into the pulse sequence, the resulting coherence β is a nonmonotonic function of D. With low noise levels, the relative width $\omega_p/\Delta\omega$ of the peak decreases, and the height H increases. Thus, the dominant effect is the regularization of the duration of the slow motion. With larger noise values, the coherence diminishes when the increasing variability of T_f becomes dominant and produces a decrease in $\omega_p/\Delta\omega$. (For very large noise amplitudes, this can lead to a decrease in H.) The optimal noise amplitude that maximizes the coherence β is in good agreement with the value obtained for system Eq. (2) (see Fig. 1).

Thus, our results show that CR is the consequence of the different dependencies of the variances of the slow and fast motions, respectively, on the input noise level. Importantly, these results are also valid for larger values of parameter a (e.g., $a=1.1$). In such cases, the optimal value of D is larger and the maximum value of β decreases with a. Recall that the system Eq. (2) is oscillatory if $a\leqslant 1$. To see if CR could also occur in the oscillatory system, we computed the coherence β as a function of the noise amplitude D for different values of the rate $1-a$ of the slow motion [the rate of the fast motion remained at the value considered above ($b=2.05$)]. The results are presented in Fig. 4. With the oscillatory system, the coherence is infinite for $D=0$ and decreases for low values of D. If a is relatively close to 1, then there exists a local maximum of β. This maximum is due to the fact that with low values of D, the variance of T_f first increases and then decreases. If the decrease of var(T_s) occurs for small values of D, then CR is obtained. However, when a is considerably smaller than 1, var(T_s) starts decreasing for large values of D for which the variations of T_f dominate the coherence of the dynamics and, as a result, the regularization induced by the decrease of var(T_s) is not noticeable.

Thus, provided that a is not too far away from the system's bifurcation value (e.g., 1), CR is present in both the excitable and oscillatory regimes. However, a is not the only important parameter for obtaining CR. The other crucial condition is that the ratio $|1-a|/b$ must be much less than 1,

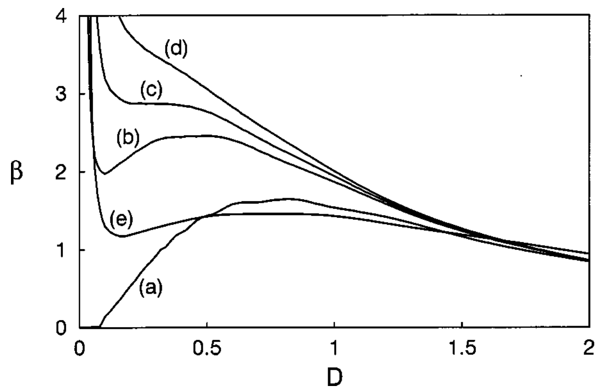

FIG. 4. Coherence β as a function of the noise amplitude D for various values of the parameters a and b: (a) $a=1.05$, $b=2.05$, (b) $a=0.85$, $b=2.05$, (c) $a=0.8$, $b=2.05$, (d) $a=0.75$, $b=2.05$, and (e) $a=0.75, b=3.42$. Note that the system is oscillatory if $a \leqslant 1$ and excitable if $a>1$.

i.e., the motion must be nonuniform. This point is illustrated by curves (d) and (e) in Fig. 4. Both curves correspond to the same low value of a $(a=0.75)$, but only for curve (e) is the ratio $|1-a|/b$ low enough so that CR is obtained. Note also that although the ratio $|1-a|/b$ for curve (b) is larger than that for curve (e), the local maximum of the coherence β is higher for curve (b) and it corresponds to a lower optimal noise level. This is because curve (b) is closer to the bifurcation. Thus, the system does not have to be in the close vicinity of the bifurcation to exhibit CR, but the CR effect is more pronounced the closer the system is to the bifurcation.

In this paper we have shown that CR in certain types of excitable and oscillatory systems is the consequence of the different effects of noise on the systems' slow and fast motions. The results we presented are not restricted to a model with a piecewise linear potential. We obtained similar results with the active rotator model [2] and an integrate-and-fire neuronal model. The CR mechanism demonstrated in this paper accounts for the CR reported in Ref. [8] and is close to the one responsible for the CR presented in Ref. [10].

The mechanism investigated here has possible implications for noisy dynamical systems of higher dimensions. In particular, our work suggests that the presence of phases of slow and fast motions in the time course of a system's variables may be an indication that an appropriate nonzero level of noise could lead to more regular dynamics. For instance, we suggest that similar CR-type effects could account for previously reported oscillations in networks of coupled excitable elements [3]. This issue will be explored in a future study.

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